Turning Complex Mathematics into Harmless Doodles

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We have developed an interactive theorem prover that uses diagrammatic representations to try and turn complex abstract mathematics into harmless doodles. Users can prove theorems by creating and reasoning with such diagrams. This is based on the idea that, for some domains (principally geometry), diagrammatic reasoning is considerably easier to understand than conventional algebraic approaches. Our system was developed for the domain of mathematical analysis (a geometric domain, but traditionally taught using a dry algebraic formalism), and recently tested on undergraduate maths students.

Diagrams in Mathematics

Diagrams are commonly used in virtually all areas of representation and reasoning. In particular – although current theorem-provers make very little use of diagrams – they are invaluable in mathematics texts. They are used in a variety of ways, including to give examples showing why a theorem is true, to give counter-examples, to explain the structure of a proof, and to prove a theorem outright. Insight is often more clearly perceived in these diagrammatic proofs than in the corresponding algebraic proofs. As Nelsen observes in "Proof without Words", in English 'to see' is often 'to understand' [2].

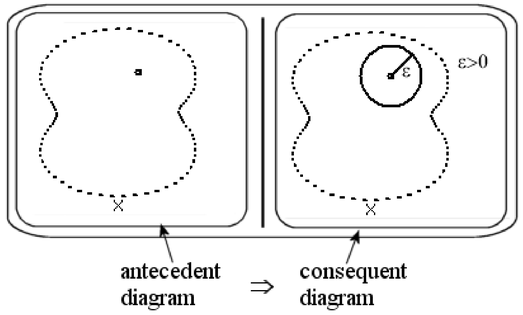
Our Domain: Mathematical Analysis

Euclidean plane geometry has always been taught using diagrammatic reasoning. Traditionally though, only algebraic proofs are allowed in the slippery realms of more abstract geometries. We have investigated using diagrams in such a domain, that of *mathematical analysis* (which underpins calculus & leads on to topology – [1] provides an excellent introduction to the subject). This is a hard domain, and even great mathematicians such as Cauchy have made mistakes in this subject. Students typically find it daunting, and we conjecture that the dry algebraic formalism used in the domain is partially responsible for these difficulties.

Dynamic Diagram Logic & the Dr.Doodle System

Often diagrammatic reasoning is presented as a question of interpreting static diagrams. We consider the process of drawing to be important, & have developed a logic for *dynamic* diagrammatic reasoning. This is implemented in an interactive theorem prover called Dr.Doodle.

Inference rules are specified using *redraw rules*, which are a visual adaptation of rewrite rules. Redraw rules are defined by an example diagram transformation; the figure below shows an example. Theorems are stated in the same manner. A proof consists of a demonstration that the theorem antecedent can always be redrawn to give the consequent diagram using an accepted set of rules.



A redraw rule for "X an open set, $x \in X \Rightarrow \exists \epsilon > 0$ s.t. $\{x' : |x' - x| < \epsilon\} \subset X$"

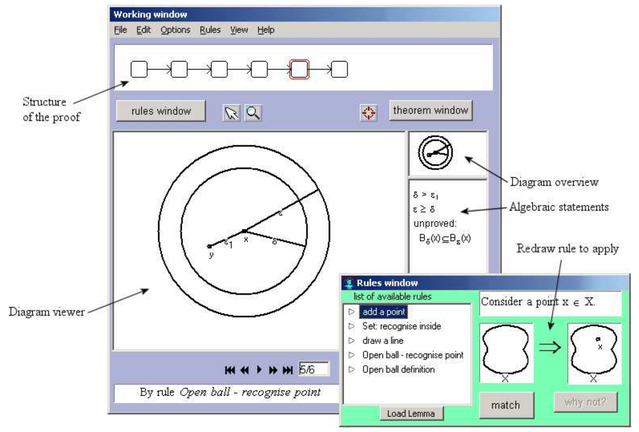
When a rule specifies creating new objects, these are typically drawn-in by the user (automated drawing, whilst interesting, requires both model-generation and aesthetic judgement to select 'good' examples). Hence diagrammatic proofs often resemble using a graphics program.

The diagrams in our system combine graphical and textual elements. They consist of example objects with conjunctions of relation statements. Disjunctions are represented using multiple diagrams. Relational statements can be represented in three different ways: *implicitly* (where the relation is true for the objects drawn, e.g. $a \in B$ for $a = \frac{1}{2}$, $B = [0, 1]$), *graphically* (using conventions such as 'a dotted border indicates an open set') or *algebraically*. These methods are used as appropriate (e.g. relations such as $A \subset B$ are usually implicit, but can also be stated algebraically if necessary).

There are many design choices in creating such a logic, including:

- How to choose and draw the example objects.
- Which relations can be represented implicitly.
- How to represent the explicit relations (e.g. arrows for function relations, but predicates for properties such as 'surjective').

There is often a trade off between simplicity, flexibility, and intuitive appeal, and our choices are tailored to the domain considered.



Screenshot of Dr.Doodle.

Experimental Evaluation

Pilot experiments were conducted to test the use of Dr.Doodle in a practical setting: teaching mathematics undergraduates. Currently Dr.Doodle only covers a fraction of the domain, but this was sufficient to run some tutorials on the concepts of *open sets* & *continuous functions*. These experiments compared performance using Dr.Doodle in algebraic and diagrammatic modes. The results show statistically significant improvements – both in exercise scores and 'efficiency' – when using diagrammatic reasoning. Informal feedback gave comments such as: "The pictures were useful for helping understand what was going on. Better than written explanations a lot of the time."

These positive results are not surprising. As the domain is a geometric one, we would expect visual representations to be useful. We conclude that diagrammatic reasoning is a useful tool in this field. However further experiments are desirable, especially as these experiments did not look at the interesting questions of *how* and *why* diagrams are useful here (and hence how general these findings are).

References

[1] T.W.Körner "A Companion to Analysis", available online at http://www.dpmms.cam.ac.uk/∼twk/

[2] R.B.Nelsen "Proofs Without Words" Mathematical Association of America, 1993.