School of Informatics, University of Edinburgh

Centre for Intelligent Systems and their Applications

On differences between the real and physical Plane: Analysis of the inside relation

by

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Technical appendix to a paper in proceedings of Diagrams 2004

Abstract :
This report is intended to be read as a technical appendix to ”On Differences Between the Real and Physical Plane” (D.Winterstein, A.Bundy & M.Jamnik, Diagrams 2004, Springer-Verlag). It gives proofs for the following two theorems: If a appears to be inside b but isn’t, then b has a closing eye structure For all star-shaped curves g, if a appears to be inside g, then a is inside g.

Keywords : formalisation diagrammatic reaoning

On differences between the real and physical Plane: Analysis of the inside relation

Daniel Winterstein

Ph.D student, Division of Informatics, Edinburgh University
Supervisors: Prof. Alan Bundy & Dr. Corin Gurr

Abstract: This report is intended to be read as an appendix to [1]. Much of it was extracted from [2] (with some adaptation). It gives proofs for the following two theorems:

- If *a* appears to be inside *b* but isn't, then *b* has a closing eye structure
- For all star-shaped curves *g*, if *a* appears to be inside *g*, then *a* is inside *g*

1 Preliminaries

Normally, the clear interior of a curve will lie inside the interior of that curve (i.e. the physical inside of a curve is smaller than the ideal inside). However, as Figure 1 shows, this is not always so.

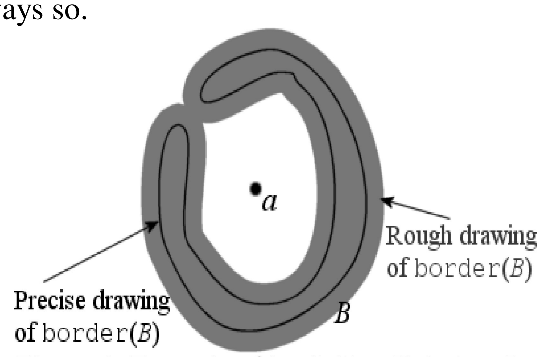

Figure 1. Example of implicit($a \in B$) *but* $a \notin B$

Lemma 1 proves that the 'closing eye' structure shown in Figure 1 is essentially the only way in which this issue can arise. The condition that basic sets are star-shaped (a generalisation of *convex*) gives one way of preventing such cases from occurring.

Assumption linking the real and physical planes

In any reasoning system, it is necessary to assume a link between the physical representations and an abstract logical representation. For example, in conventional logic we assume that the reasoner can reliably convert symbols on a page (the physical representation) into an array of symbols in the logic, and in automated reasoning we assume that computers will behave as intended. However for diagrammatic reasoning, the link between physical and logical representations is considerably more complex.

Here we assume a model in which diagrams are created by attempting to physically draw objects from $\mathbb{R}^2$. We assume that the surface on which diagrams are drawn obeys euclidean plane geometry up to detectable differences. This is uncontroversial. The true geometry of physical space is unknown, but probably not euclidean. However flat surfaces certainly appear to be euclidean – that is, they are identical to a euclidean surface to the naked eye – which is all we are assuming.

Accuracy

There are several ways in which physical representations are inaccurate when compared with representations situated in $\mathbb{R}^2$: Drawing tools have a certain width; they cannot draw infinitely fine lines. They also have a limited precision, and will inevitably make small errors. We must also consider the measuring tools used for reading the diagram (typically human eyesight), which also have limited precision.

For our purpose (where we are not concerned with the mechanics of the drawing process or the internal details of the observation process), it makes sense to combine the effects of these disparate factors. We model this by introducing one measure ε for accuracy, which combines the width and precision of drawing tools, and the precision of the measuring tools. ε will therefore vary depending on the drawing system used, and we do not fix its value here.

2 Notation and Definitions

2.0.1 Notation

Power sets: If X is a set, write $P(X)$ for the power set of X
Flattening function: Given $A \subset P(S)$, define $\mathrm{U}(A) = \cup_{a \in A} a$
Paths: If x,y are points in $\mathbb{R}^2 \cup \{\infty\}$, let $p{:}x \to y$ denote that p is a path from x to y (i.e. p is a function, $p{:}[0,1] \to \mathbb{R}^2$, $p(0)=x$, $p(1)=y$). Given a simple path p passing through points x,y, let $p{:}x \to y$ denote the section of path starting at x and ending at y.
Balls: Write $B_r(x)$ for $\{x' : |x'-x|<r\}$ (the open ball of radius r around x)
Closure: If X is a set, write $[X]$ for the closure of X (this unusual notation is forced on us by formatting limitations of the word processor used).

2.0.2 Definitions

Definition 1: Drawing Surface
Given a physical drawing surface $\mathcal{S}$ with a fixed x-y orientation and scale, assume that $\mathcal{S}$ is indistinguishable from rectangle S in $\mathbb{R}^2$ upto detectable differences.

We can now analyse the effects of producing physical drawings of curves in $\mathbb{R}^2$ as a mapping between objects in S.

Definition 2: Drawing Function
Given a drawing process with accuracy ε, define the drawing function $\mathbb{D}:S\rightarrow P(S)$ by $\mathbb{D}(x) = B_\varepsilon(x)$. We extend $\mathbb{D}$ to give $\mathbb{D}:P(S)\rightarrow P(S)$ by $\mathbb{D}(A) = \cup_{a\in A}\,\mathbb{D}(a)$

Definition 3: Admissible Curves
A curve $g\subset\mathbb{R}^2$ is *admissible* if g is a non-intersecting closed curve, $g\subset S$, g does not touch the border of S

Definition 4: Interior
Given a set $A\subset S$, let `interior`(A) be the set $\{x\in S\backslash A : \forall$ paths $p:x\rightarrow\infty, p\cap A\neq\emptyset\}$

Note: The Jordan Curve Theorem states that if g is an admissible curve, then `interior`(g) is homeomorphic to $\{x\in\mathbb{R}^2 : |x|<1\}$

Definition 5: Clear Interior
Given a set $A\subset S$ let `clear-interior`(A) be the set $\{x\in S\backslash A : \mathbb{D}(x)\subset$`interior`$(\mathbb{D}(A))\}$

Definition 6: Observable relations
Given an admissible curve g and a point $a\in S$, we say "a inside g" is observable if $\mathbb{D}(a)\subset$`clear-interior`(g)
Given admissible curves g, f, we say "f inside g" is observable if $\mathbb{D}(f)\subset$`clear-interior`(g)

Definition 7: Touching Points
$x,y\in S$ are *touching points* if $\mathbb{D}(x)\cap\mathbb{D}(y)\neq\emptyset$

Definition 8: Star-shaped (a generalisation of *convex*)
A set X is *star-shaped* if $\exists\ c\in X$ such that $\forall x\in X$, the line cx is in X
We will call a curve g star-shaped if its interior is star-shaped.

The next definition is only needed for the proof:

Definition 9: ε-patch set
Given a set $X\subset S$, $A\subset P(S)$ is an *ε-patch set* for X if $\forall a\in A$, $a\neq\emptyset$, $a=B_\varepsilon(y)\cap\mathbb{D}(X)\backslash X$, for some $y\in X$. Call $\{y : B_\varepsilon(y)\cap\mathbb{D}(X)\backslash X \in A\}$ the *generating points* for A.

Lemma 1: Given an admissible curve b and a point $a\notin$`interior`(b) but `observable`$(a$ `inside` $b)$. Then $\exists$ points $z_1, z_2\in b$, line $l=z_1z_2$ such that z_1, z_2 are touching points and $a\in$`interior`$(b\cup l)$

Proof:
Let B = interior$(b)\cup b$. By the Jordan Curve Theorem, B is connected.
Let M be a minimal ε-patch-set for B such that $a\in$`interior`$(B\cup \mathrm{U}(M))$ (noting that

such an M always exists and is finite)

Let $B^+ = B \cup \mathrm{U}(M)$

$a \in \texttt{interior}(B^+) \Rightarrow \exists$ simple closed curve $g \subset B^+$ such that $a \in \texttt{interior}(g)$

Note that M minimal $\Rightarrow g \cap m \neq \emptyset \ \forall m \in M$ for any such g

Also, we can choose g such that it passes through the generating points for M, and $g \cap B \cap [m] \neq \emptyset \ \ \forall m \in M$

Claim 2: $\mathrm{U}(M)$ is connected

Suppose false: $\Rightarrow$ we can split M into M_1, M_2 such that $\mathrm{U}(M_1)$, $\mathrm{U}(M_2)$ are not connected

$\mathrm{U}(M_i)$ not connected, g a loop $\Rightarrow \ \exists$ points $x,y \in g$□$\mathrm{U}(M)$ such that $g{:}x \rightarrow y$ passes through M_1, $g{:}y \rightarrow x$ passes through M_2

Let $g_1 = g{:}x \rightarrow y$, $g_2 = g{:}y \rightarrow x$

B connected, $x,y \in B \Rightarrow \exists$ curve $g' \subset G$, $g'{:}x \rightarrow y$

Now $g_i \cup g'$ are closed curves, $g_1 \cup g' \cap M_2 = \emptyset$ and $g_1 \cup g' \cap M_2 = \emptyset$

Also $\texttt{interior}(g) \subset (\cup_i \texttt{interior}(g_i \cup g') \cup g')$

Hence $a \in \texttt{interior}(g_1 \cup g')$ or $\texttt{interior}(g_2 \cup g')$.

WLOG say $a \in \texttt{interior}(g_1 \cup g')$

But $a \in \texttt{interior}(g_1 \cup g')$, M minimal $\Rightarrow (g_1 \cup g') \cap m \neq \emptyset \ \forall m \in M$. So this contradicts the minimality of M, hence M must be connected as claimed. Figure 2 gives an example illustrating this situation.

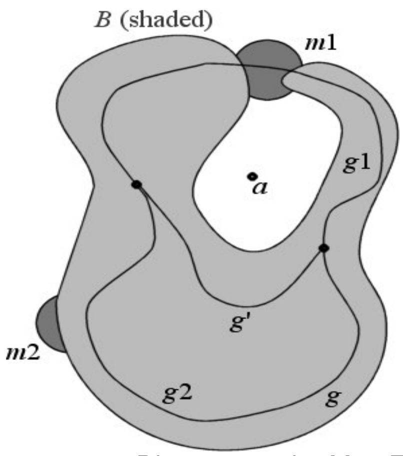

Figure 2. Disconnected M. The contradiction in this example is that M_2 is not necessary for $a \in \texttt{interior}(B+)$

Claim 3: $|M| < 3$

Suppose false, and let m_1, m_2, m_3 be distinct ε-balls in M

Let x_i be points in $B \cap g \cap [m_i]$

Let g_1 be the portion of g joining x_1 to x_2, and g_2 the portion of g joining x_2 to x_3

g a simple curve $\Rightarrow \ g_1, g_2$ are disjoint except for x_2

Now B connected $\Rightarrow \exists$ curves $g_1', g_2' \subset B$ such that g_1' joins x_1 to x_2, g_2'

joins x_2 to x_3.

$g_1 \cup g_1'$ is a closed curve. Moreover, we have $a \in \mathtt{interior}(g_1 \cup g_1')$, since otherwise we would have $a \in \mathtt{interior}(g_1' \cup g_2 \cup g_3)$, which would contradict the minimality of M

Similarly $a \in \mathtt{interior}(g_2 \cup g_2')$

So $a \in \mathtt{interior}(g_1 \cup g_1') \cap \mathtt{interior}(g_2 \cup g_2')$

This is illustrated in Figure 3.

Now suppose g_1', g_2' intersect at a point y

Then let f be the path $f: x_2 \to y \to x_2$ formed from g_1', g_2'

But then $a \in \mathtt{interior}(f)$, which implies $a \in \mathtt{interior}(B)$.

Contradiction, hence $|M|<3$

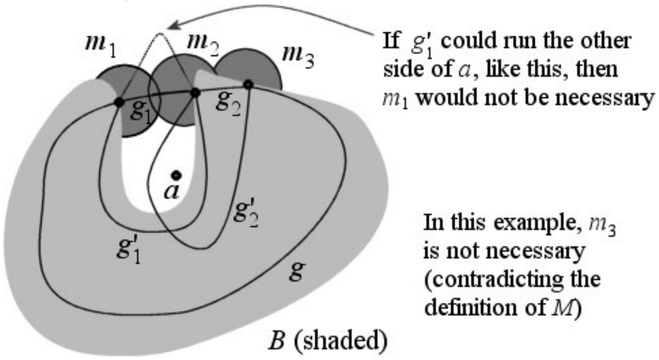

Figure 3. |M|>2

Therefore $g_1' \cap g_2' = \{x_2\}$.

But then consider the path $f': x_2 \to x_3 \to x_1 \to x_2$ made from g_2', $g \backslash g_i$, g_1' respectively. This has $a \in \mathtt{interior}(f')$ and $f' \subset B^+ \backslash m_i$ – contradicting the minimality of M.

Now $|M|<3$, M connected $\Rightarrow |M|=1$ *or* $|M|=2$ (since $|M|=0 \Rightarrow a \in B$)

If $|M|=1$, let z_1, z_2 be the first and last points of g in $[U(M)]$.

Then $z_1, z_2 \in B$. Let l be the line z_1z_2, and note that $l \subset B^+$. Now points z_1, z_2 can be linked by a curve $g' \subset B$, and $g' \cup l$ will be a closed curve such that $a \in \mathtt{interior}(g' \cup l)$. Hence $a \in \mathtt{interior}(b \cup l)$ as required.

Otherwise $|M|=2$, and since M is connected, it was generated by touching points $z_1, z_2 \in B$. Let l be the line z_1z_2. Points z_1, z_2 can be linked by a curve $g' \subset B$, and $g' \cup l$ will be a closed curve such that $a \in \mathtt{interior}(g' \cup l)$. Hence $a \in \mathtt{interior}(b \cup l)$ as required.

□

Lemma 2: Given a triangle Δxyz such that x,y are touching points, then there are no points that are clearly inside Δxyz.

Proof:

Line xy is the longest line from a point in line xz to a point in line yz.

x,y touching points $\Rightarrow |xy|<2\varepsilon$

But to be clearly inside Δxyz, a point needs its minimum distance to the triangle edges to be at least 2ε. Hence no such points exist.

Lemma 3: For all star-shaped admissible curves g, a inside $g \Rightarrow a \in \texttt{interior}(g)$

Proof:
Suppose false
Let $I = \texttt{interior}(g)$
By Lemma 1, $\exists$ points $x,y \in g$ such that $a \in \texttt{interior}(g \cup xy)$
All basic sets are star-shaped, hence $\exists$ point $c \in I$ such that lines xc, $yc \subset [I]$
Let $f \subset I \cup xy$ be a simple closed curve such that $a \in \texttt{interior}(f)$
Suppose $a \notin \Delta xyc$. But then $a \in \texttt{interior}((f \backslash xy) \cup xc \cup yc) \subset I$. This contradicts $a \notin I$, hence $a \in \Delta xyc$. But by Lemma 2 there are no such points, hence we are done.

Corollary 3.1: For all simple closed curves f, g such that g is star-shaped,
f inside $g \Rightarrow f \subset \texttt{interior}(g)$

3 References

[1] D.Winterstein, A.Bundy & M.Jamnik "On Differences Between the Real and Physical Plane" submitted to the Diagrams 2004 conference.

[2] D.Winterstein "Diagrammatic Reasoning in a Continuous Domain" forthcoming Ph.D. thesis, Edinburgh University, 2004.